Graph Decompositions and Moore Graphs: Two Applications of Spectral Graph Theory

MATH 0746 Thesis

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Abstract

This paper introduces readers to the branch of mathematics known as spectral graph theory by discussing two applications of graph eigenvalues, both of which happen to involve the Petersen graph. First, we provide an elegant proof of the fact that the complete graph on ten vertices cannot be decomposed into three copies of the Petersen graph. Second, we introduce the idea of a Moore graph. After showing that three separate definitions of Moore graphs are equivalent, we use graph eigenvalues to show that for a diameter of 2 or a girth of 5, Moore graphs may only be $3, 7,$ or 57-regular. We prove the existence and uniqueness of the 3-regular case (the Petersen graph), contextualize the 7-regular case, and acknowledge the still open 57-regular question. Our proofs rely on results from linear algebra, illustrating the power of eigenvalues and vector spaces in solving problems that would otherwise be tedious and messy, if not impossible.

Contents

1 Introduction

Spectral methods are a powerful and relatively new set of tools in graph theory, which involve the eigenvalues and eigenvectors of certain matrix representations of graphs. In this paper, we introduce readers to this branch of mathematics by discussing two well-known applications – an attempted graph decomposition and a problem in extremal graph theory. In order to discuss these results, we must first define some terminology.

1.1 Basic Terminology

Definition 1.1.1 – Adjacency Matrix [12]: Let G be a graph and label its vertices $\{1, 2, ..., n\}$. The adjacency matrix of G is the $n \times n$ matrix A with:

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and vertex } i \text{ is adjacent to vertex } j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.1.2 – Graph Eigenvalues and Eigenvectors [17]: The eigenvalues and eigenvectors of a graph's adjacency matrix.

Definition 1.1.3 – Graph Spectrum [5]: The set of all eigenvalues of a graph.

Here we note that there are alternative definitions of graph eigenvalues, eigenvectors, and spectra which are based on different matrix representations of a graph. The Laplacian matrix, in particular, has gained popularity in recent years due to the way it relates to other graph invariants [5]. While alternative matrix representations are advantageous for certain applications, the results that we discuss were originally proven and are typically discussed using the adjacency matrix. For further discussion of alternative approaches, we refer readers to other sources such as *Spectral Graph Theory* by Fan Chung.

1.2 Why Does Spectral Graph Theory Matter?

Graph theory is a relatively new branch of mathematics – the term *graph*, as we use it here, did not appear in print until 1878 [2]! The field of spectral graph theory is even younger, yet its importance cannot be overstated.

Google's PageRank algorithm involves the Perron-Frobenius eigenvector of the web graph, making the applications of spectral graph theory as ubiquitous as the Google search [4, 3]. Spectral methods have generated billions of dollars for Google and laid the foundation for the way we use the internet today.

While most applications of graph eigenvalues and eigenvectors are lesser known than the Google search, they are nonetheless important. Spectral methods are a rapidly growing area in graph theory, and the two examples discussed in this paper provide a brief glimpse of their elegance and utility.

2 Can K_{10} be Decomposed into Three Copies of the Petersen Graph?

2.1 Problem Statement

Recall the Petersen graph and the complete graph on ten vertices, K_{10}. The Petersen graph contains 10 vertices, each of degree 3, and is one of the most famous graphs in graph theory. K_{10} contains 10 vertices, each pair of which share an edge, making each vertex in K_{10} have degree 9.

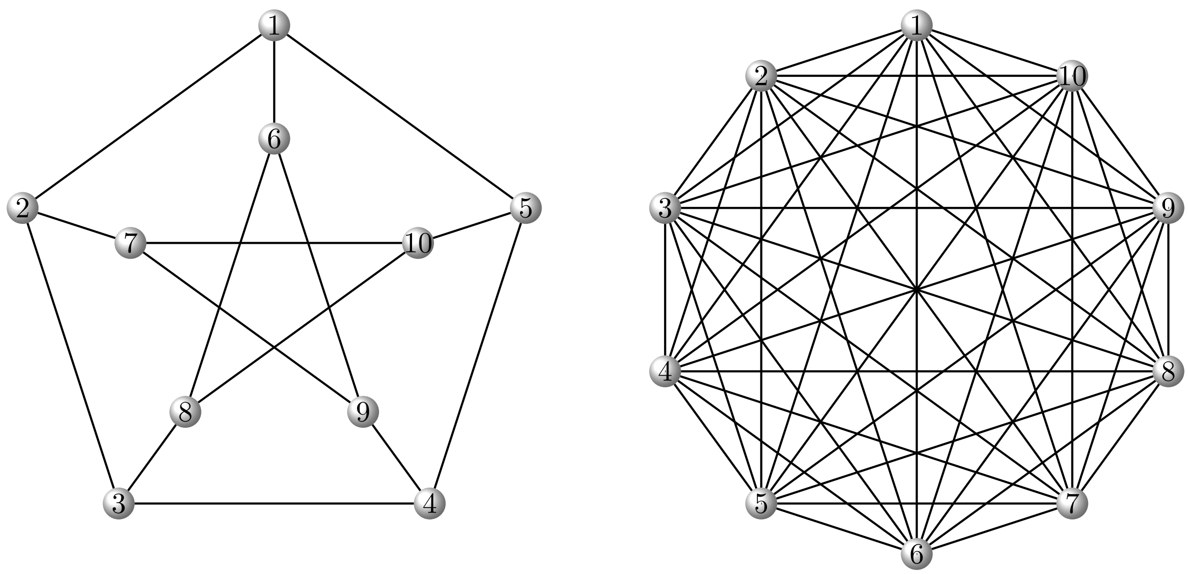


Figure 1: The Petersen graph on the left and K_{10} on the right.

Since the Petersen graph is 3-regular, K_{10} is 9-regular, and both graphs contain ten vertices, it seems arithmetically feasible that K_{10} could be decomposed into three copies of the Petersen graph. In this section, we will prove that such a decomposition is in fact not possible using the eigenvalues of their adjacency matrices. First, we need to define a few terms and recall some results from linear algebra and graph theory.

2.2 Terms and Results from Linear Algebra and Graph Theory

Definition 2.2.1 – Permutation Matrix [16, 17]: A permutation matrix is a matrix that can be obtained by performing row and column interchanges on the identity matrix. Permutation matrices are all zeros except for a one in every row and a one in every column.

Definition 2.2.2 – Similar Matrices [11]: Two $n \times n$ matrices A and B are similar if there exists some invertible matrix P such that $B = P^{-1}AP$, or equivalently, $A = PBP^{-1}$.

Theorem 2.2.3 [11]: If $n \times n$ matrices are similar, they have the same set of eigenvalues with the same multiplicities.

This is a result from linear algebra whose proof can be readily found in a linear algebra textbook.

Lemma 2.2.4 [15, 16, 6]: Isomorphic graphs have the same set of graph eigenvalues with the same multiplicities.

Proof. Consider two isomorphic graphs G and H with adjacency matrices A_G and A_H. If the vertices of G and H are labelled in the same way, then the adjacencies of vertices with the same label are the same, and thus the adjacency matrices A_G and A_H are identical. Hence, the eigenvalues and their multiplicities of G and H are the same.

Suppose that G and H have different vertex labellings. Since G and H are isomorphic, we could obviously relabel the vertices of H to match those of G, producing a new adjacency matrix for H identical to that of G. For the sake of internal consistency, however, we seek to show that the adjacency matrix corresponding to H's original labelling has the same eigenvalues as that of G.

Since H is isomorphic to G, we know that there exists some relabelling – or permutation – of H's vertices that matches the labelling of G's vertices. In terms of adjacency matrices, this would amount to changing the order of A_H's rows and columns such that they match the rows and columns of A_G.

Recall from linear algebra that performing elementary row operations on a given matrix A is equivalent to performing those same row operations on the identity matrix I and right-multiplying that matrix by A. For this reason, we can construct a permutation matrix P that reorders the rows of A_H by multiplication: PA_H. Similarly, we can reorder the columns by right-multiplying by P^{-1}. Thus $A_G = PA_HP^{-1}$ for some permutation matrix P.

Since we can write $A_G = PA_HP^{-1}$, we know that A_G and A_H are similar matrices by definition 2.2.2. Since A_G and A_H are similar matrices, they must have the same set of eigenvalues with the same multiplicities by theorem 2.2.3.

□

Definition 2.2.5 – Orthogonal Complement [11]: Let W be a subspace of $\mathbb{R}^n$. The set of all vectors $\vec{z}$ orthogonal to W is known as the orthogonal complement of W, denoted $W^\perp$.

Definition 2.2.6 – Sum and Intersection of Subspaces [10]: Let V be a vector space with subspaces U and W. The sum of U and W is defined as $U + W = \{\vec{u} + \vec{w} | \vec{u} \in U, \vec{w} \in W\}$. The intersection of U and W is defined as $U \cap W = \{\vec{v} \in U, \vec{v} \in W\}$.

Theorem 2.2.7 – Dimension of Sum of Subspaces [10]: Let V be a vector space with finite-dimensional subspaces U and W. The sum of U and W is also finite-dimensional with $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

2.3 Proof of Theorem

Theorem 2.3.1 [12]: K_{10} cannot be decomposed into three copies of the Petersen graph.

Proof. Assume, for contradiction, that K_{10} can be decomposed into three copies of the Petersen graph. We label these Petersen graphs P, Q, and R, and we label their adjacency matrices as A_P, A_Q, and A_R.

Notice that the adjacency matrix of K_{10} is $J_{10} - I_{10}$. For clarity on notation, J_{10} is the 10×10 matrix of all one's and I_{10} is the 10×10 identity matrix, so $J_{10} - I_{10}$ is the 10×10 matrix with ones everywhere except the main diagonal, which has zeros. This is the adjacency matrix of K_{10} because every pair of distinct vertices are adjacent but there are no loops in K_{10}.

If our assumption is correct, then we can also express the adjacency matrix of K_{10} as the sum of A_P, A_Q, and A_R. We will eventually find our contradiction by manipulating the resulting equality:

$$A_P + A_Q + A_R = J_{10} - I_{10}$$

To do this, we first need to determine the eigenvalues of the Petersen graph.

Note that for a given numbering of the vertices of K_{10} such as the one given in Figure 1, the subgraphs P, Q, and R will have different adjacency matrices since they cannot share edges. However, since P, Q, and R are isomorphic graphs, by lemma 2.2.4, they have the same set of eigenvalues and the same dimensions of their eigenspace. Thus, by finding the eigenvalues of one vertex labelling of the Petersen graph, we have the eigenvalues for all vertex labellings, including those of P, Q, and R.

We construct the adjacency matrix A_{Petersen} of one labelling of the Petersen graph by manually inspecting the labelling of the Petersen graph given in Figure 1.

$$A_{\text{Petersen}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Using the programming language, R, we calculate the eigenvalues of A_{Petersen} to be 3 (with a multiplicity of 1), 1 (with a multiplicity of 5), and -2 (with a multiplicity of 4). We proceed by investigating the eigenvalue of 1 in particular. By the definitions of eigenvalues and eigenvectors, we know that $A_{\text{Petersen}}\vec{x} = \lambda\vec{x} = \vec{x}$ for some vectors $\vec{x}$ letting $\lambda = 1$. In R, we perform Gaussian elimination on the augmented matrix corresponding to $A_{\text{Petersen}}\vec{x} = \vec{x}$, resulting in the following solution.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} x_6 - x_8 - x_9 \\ x_7 - x_9 - x_{10} \\ x_8 - x_6 - x_{10} \\ x_9 - x_6 - x_7 \\ x_{10} - x_7 - x_8 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = x_6 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_8 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_9 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_{10} \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Recall from definition 2.2.5 that the set of all vectors orthogonal to $\text{span}\{\vec{1}\}$ is known as the orthogonal complement of $\text{span}\{\vec{1}\}$, denoted $\text{span}\{\vec{1}\}^{\perp}$. Notice that there are two 1's and two -1's in each of the five eigenvectors corresponding to the eigenvalue of 1, so the dot product of each eigenvector with $\vec{1}$ is 0. By definition, each of these eigenvectors are therefore orthogonal to $\vec{1}$ and thus orthogonal to $\text{span}\{\vec{1}\}$. Since each eigenvector corresponding to the eigenvalue $\lambda = 1$ is orthogonal to $\text{span}\{\vec{1}\}$, their span is also orthogonal to $\text{span}\{\vec{1}\}$ and thus all vectors in the eigenspace of $\lambda = 1$ are contained in $\text{span}\{\vec{1}\}^{\perp}$. Also note that $\text{span}\{\vec{1}\}^{\perp}$ is 9-dimensional because we are working in $\mathbb{R}^{10}$ and $\text{span}\{\vec{1}\}$ is 1-dimensional.

Since the eigenspace of the Petersen graph is contained in $\text{span}\{\vec{1}\}^{\perp}$, and the eigenvalues and eigenspaces of isomorphic graphs are the same, we know that the eigenspaces of A_P and A_Q associated with eigenvalue $\lambda = 1$ are both contained in $\text{span}\{\vec{1}\}^{\perp}$. Recall that the eigenspace corresponding to some matrix A and some eigenvalue λ is the same as the kernel of $A - \lambda I_n$. For this reason, the eigenspaces of A_P and A_Q corresponding to eigenvalue $\lambda = 1$ are the same as the kernel of $A_P - I_{10}$ and $A_Q - I_{10}$, and thus the kernels of $A_P - I_{10}$ and $A_Q - I_{10}$ are both contained in $\text{span}\{\vec{1}\}^{\perp}$. Since the kernels of $A_P - I_{10}$ and $A_Q - I_{10}$ are subspaces of $\mathbb{R}^{10}$, they must contain the zero vector and be closed under vector addition and scalar multiplication. Since the kernels of $A_P - I_{10}$ and $A_Q - I_{10}$ are contained in $\text{span}\{\vec{1}\}^{\perp}$, contain the zero vector, and are closed under vector addition and scalar multiplication, by the definition of subspace, they are subspaces of $\text{span}\{\vec{1}\}^{\perp}$ [11].

Since $\text{kern}(A_P - I_{10})$ and $\text{kern}(A_Q - I_{10})$ are both subspaces of $\text{span}\{\vec{1}\}^{\perp}$, their sum is also a subspace of $\text{span}\{\vec{1}\}^{\perp}$. Since their sum is a subspace of $\text{span}\{\vec{1}\}^{\perp}$, the dimension of their sum is bounded above by the dimension of $\text{span}\{\vec{1}\}^{\perp}$; that is, $\dim(\text{kern}(A_P - I_{10}) + \text{kern}(A_Q - I_{10})) \leq 9$. Applying theorem 2.2.7, we have $\dim(\text{kern}(A_P - I_{10})) + \dim(\text{kern}(A_Q - I_{10}))$ - $\dim(\text{kern}(A_P - I_{10})$ $\cap$ $\text{kern}(A_Q - I_{10})) \leq 9$. Since the dimension of the eigenspaces of A_P and A_Q for $\lambda = 1$ are both 5, we can write this as 10 - $\dim(\text{kern}(A_P - I_{10}) \cap \text{kern}(A_Q - I_{10})) \leq 9$. Applying arithmetic, we have $\dim(\text{kern}(A_P - I_{10}) \cap \text{kern}(A_Q - I_{10})) \geq 1$.

Since $\dim(\text{kern}(A_P - I_{10}) \cap \text{kern}(A_Q - I_{10})) \geq 1$, we know that $\text{kern}(A_P - I_{10})$ and $\text{kern}(A_Q - I_{10})$ have at least one non-zero vector, call it $\vec{x}$, in common. Noting that $J_{10}\vec{x} = \vec{0}$ because $\vec{x}$ is orthogonal to $\vec{1}$, we calculate:

$$J_{10} - I_{10} = A_P + A_Q + A_R$$
$$A_R = J_{10} - I_{10} - A_P - A_Q$$
$$A_R\vec{x} = (J_{10} - I_{10} - A_P - A_Q)\vec{x}$$
$$A_R\vec{x} = (J_{10} - I_{10} - (A_P - I_{10}) - (A_Q - I_{10}) - 2I_{10})\vec{x}$$
$$A_R\vec{x} = J_{10}\vec{x} - I_{10}\vec{x} - (A_P - I_{10})\vec{x} - (A_Q - I_{10})\vec{x} - 2I_{10}\vec{x}$$
$$A_R\vec{x} = \vec{0} - \vec{x} - \vec{0} - \vec{0} - 2\vec{x}$$
$$A_R\vec{x} = -3\vec{x}$$

By the definition of eigenvalues and eigenvectors, this implies that -3 is an eigenvalue of subgraph R. However, earlier we calculated the eigenvalues of the Petersen graph directly from its adjacency matrix, and -3 was not one of its eigenvalues! We have therefore found a contradiction, so we reject our assumption that K_{10} can be decomposed into three copies of the Petersen graph, and we are done.

□

3 Moore Graphs

In 1960, Hoffman and Singleton published a landmark paper in spectral graph theory entitled *On Moore Graphs with Diameters 2 and 3* [8]. Their paper was one of the first to apply spectral methods to problems in extremal graph theory and has become a widely referenced piece of mathematics. In this section, our main goal is to break their arguments down into a fashion that is more comprehensible to the undergraduate reader.

3.1 Three Definitions of a Moore Graph

Definition 3.1.1 – Distance [17]: If a path exists between vertices u and v in a graph G, then the distance from u to v, denoted $d(u, v)$, is the number of edges in the shortest u, v path. If a path does not exist between vertices u and v, then $d(u, v) = \infty$.

Definition 3.1.2 – Diameter [17]: The diameter of a graph G is $\max\{d(u, v)\}$ amongst all pairs of vertices $u, v \in G$.

Scenario 3.1.3 – Degree/Diameter Problem [13]:

For $\Delta, D \in \mathbb{N}$, consider a graph G with maximum degree Δ and diameter $\leq D$. What is the largest possible number of vertices $n_{\Delta,D}$ in such a graph? In this section, we establish an upper bound on this quantity.

The scenario is trivial if $\Delta = 1$, as there can only be two vertices in a connected graph with a maximum degree of 1, and all graphs with finite diameter must be connected. For this reason, we assume $\Delta \geq 2$. In such a graph, let one vertex, v, be distinguished, and let n_i be the number of vertices at distance i from vertex v for $0 \leq i \leq D$. Since G has a maximum degree of Δ and each vertex at distance $i \geq 1$ must be adjacent to a vertex at distance $i-1$ from v, each vertex at distance i is adjacent to at most $\Delta - 1$ vertices at distance $i + 1$ from v. It follows that $n_{i+1} \leq (\Delta - 1)n_i$ for all i such that $1 \leq i \leq D - 1$.

On the other hand, since there is no distance level prior to the distinguished vertex v, the distinguished vertex may be adjacent to up to Δ vertices in $i = 1$, so we have $n_1 \leq \Delta$. Since $n_1 \leq \Delta$, the number of vertices at distance 2 from v is $n_2 \leq (\Delta - 1)n_1 \leq (\Delta - 1)\Delta$, so the number of vertices at distance 3 from v is $n_3 \leq (\Delta - 1)n_2 \leq (\Delta - 1)^2\Delta$, and so on such that the number of vertices at distance i from v is $n_i \leq \Delta(\Delta - 1)^{i-1}$ for $1 \leq i \leq D$.

Now that we have an upper bound on the number of vertices at each possible distance from v, we may return to our question regarding the largest possible number of vertices in G, $n_{\Delta,D}$. Summing the maximum number of vertices at each distance level, we have:

$$n_{\Delta,D} = \sum_{i=0}^{D} n_i \leq 1 + \Delta + \Delta(\Delta - 1) + ... + \Delta(\Delta - 1)^{D-1}$$

$$= 1 + \Delta(1 + (\Delta - 1) + ... + (\Delta - 1)^{D-1}).$$

In a situation where this upper bound is met, then G has a subgraph of the form illustrated in Figure 2. Specifically, this subgraph is a tree T of height D, where v is adjacent to Δ vertices at the next level, and all the vertices at levels 1 through $D - 1$ are adjacent to $\Delta - 1$ vertices at the next level. There may be no other edges from vertices in levels 0 through $D - 1$ in a graph G achieving this upper bound, as we have assigned all of these vertices the maximum degree Δ in the graph. For this reason, any additional vertices must be adjacent to the vertices in level D. However, the existence of any such vertices would make the diameter of G greater than D, so they must not exist. On other other hand, there must be edges connecting vertices in the topmost level. Otherwise, two vertices in the topmost level would be a distance of $2D$ from each other.

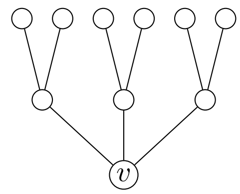


Figure 2: The subgraph T for the scenario $\Delta = 3, D = 2$.

Definition 3.1.4 – Moore Graph Definition 1 [13]: A graph G with maximum degree Δ and diameter $\leq D$ is called a Moore graph if the number of vertices in G achieves the upper bound established in Scenario 3.1.3. That is, G is a Moore graph if it contains $n_{\Delta,D} = 1 + \Delta(1 + (\Delta - 1) + ... + (\Delta - 1)^{D-1})$ vertices.

Definition 3.1.5 – Girth [12]: The girth of a graph G is the length of its shortest cycle.

Scenario 3.1.6 – Degree/Girth Problem [12]:

Consider a graph G with minimum degree $\delta \geq 3$ and girth $g \geq 4$. What is the smallest possible number of vertices $n_{\delta,g}$ in such a graph?

Assume $g = 2k + 1$ is odd for some $k \in \mathbb{N}$. In such a graph, let one vertex, v, be distinguished, and consider two paths of length k starting at u. While these two paths could share a few edges at their start, they must eventually branch and not rejoin. If they were to rejoin, then these two paths would form a cycle of length less than or equal to $2k$. For this reason, G contains a subgraph such as the one shown in Figure 2. More specifically, this subgraph is a tree T of height k, where v is adjacent to δ vertices at the next level, and all the vertices at levels 1 through $k - 1$ are adjacent to $\delta - 1$ vertices at the next level. There may be additional vertices and edges in G, but at a bare minimum, G has a subgraph of this form due to its degree and girth constraints.

By counting the number of vertices at each level of T, we can easily see that the total number of vertices in T is $1 + \delta + \delta(\delta - 1) + ... + \delta(\delta - 1)^{k-1}$. Since G must include at least as many vertices as T, we have:

$$\begin{aligned} n_{\delta,g} &\geq 1 + \delta + \delta(\delta - 1) + ... + \delta(\delta - 1)^{k-1} \\ &= 1 + \delta(1 + (\delta - 1) + ... + (\delta - 1)^{k-1}). \end{aligned}$$

Definition 3.1.7 – Moore Graph Definition 2 [12]: A graph G with minimum degree δ and girth $g = 2k + 1$ is called a Moore graph if the number of vertices in G achieves the lower bound established in Scenario 3.1.6. That is, G is a Moore graph if it contains $n_{\delta,g} = 1 + \delta(1 + (\delta - 1) + ... + (\delta - 1)^{k-1})$ vertices.

There is another bound for graphs of even girth g, and graphs that achieve that bound are known as generalized polygons, but we will not consider such graphs in this paper.

Definition 3.1.8 – Moore Graph Definition 3 [7]: If a graph G has diameter D and girth $g = 2D + 1$, then G is known as a Moore graph.

Lemma 3.1.9 [7]: A graph with diameter D has girth g at most $2D + 1$.

This lemma is critical in our proof of the fact that the three definitions of Moore graphs are equivalent. However, before we can prove the equivalence of our definitions, we need to show that Moore graphs, according to definition 3.1.8, are regular.

3.2 All Moore Graphs are Regular

For the entirety of section 3.2, let G be a graph with diameter $D \geq 1$ and girth $g = 2D + 1$. Notice that G is a Moore graph by definition 3.1.8. In this section, we prove that all graphs satisfying definition 3.1.8 of a Moore graph are regular.

Definition 3.2.1 – Antipodal [14]: Let the diameter of a graph G be D. We say that two vertices u and v in G are antipodal if $d(u, v) = D$, and that the antipodes of a vertex w are all vertices antipodal to w.

Lemma 3.2.2 [14]: Antipodal vertices in G are of equal degree.

Proof. Let a and b be antipodal vertices in G. Since G has girth $g = 2D + 1$, there must be a unique path of length D connecting a and b; if there were two such paths, then there would exist a cycle of length less than or equal to $2D$. Let c be the vertex on this unique path that is adjacent to b and a distance of $D - 1$ from a.

In this paragraph, we justify the following claim: all vertices adjacent to b other than c are antipodal to a. Let e be one such vertex. Since the diameter of G is D, there must be some path from a to e of length D or less. This path must not include the edge $e - b$, as this would violate the fact that a and b are antipodal. Knowing this, if such an a, e-path included some vertex f on the a-c-b-e path, then there would be a cycle of length less than $2D + 1$, since the distance from f to b is less than D and the distance from f to e is less than D and the distance from b to e is 1. Because the existence of a cycle of length less than $2D + 1$ violates the known girth of G, we conclude that the aforementioned a-e path and the a-c-b-e path have distinct vertices (other than a and e); in other words, there exists some a-c-b-e cycle. The distance from a to c is $D - 1$, the distance from c to b is 1, and the distance from b to e is 1, making the total distance along the edges from a to c to b to e sum to $D + 1$. If our a, e-path on distinct edges from those on the a-c-b-e path included less than D edges, than the a, c, b, e cycle would have less than $2D + 1$ edges, violating the girth of G. Thus all vertices adjacent to b other than c are indeed antipodal to a.

Let d_b denote the degree of vertex b. By the logic above, there are $d_b - 1$ vertices such as e, each of which generates a cycle of $2D + 1$ vertices that contains a, b, and the specified vertex. Each of these $d_b - 1$ cycles are unique because, as argued in the first paragraph in the proof of this lemma, there can be only one path of length D between antipodal vertices, such as a and e, and each of these $d_b - 1$ vertices are different from each other. Additionally, note that any cycle of length $2D + 1$ on vertices a and b must contain one vertex adjacent to b and antipodal to a in order for the cycle to exist. Since there are $d_b - 1$ vertices like e, each of which is contained in a singular $2D + 1$ length cycle on vertices a and b, and such a cycle cannot exist without a vertex like e, we conclude that the number of cycles of length $2D + 1$ on vertices a and b is $d_b - 1$.

The very same logic with the roles of a and b swapped illustrates that the number of cycles of length $2D + 1$ on vertices a and b is $d_a - 1$, implying that $d_a - 1 = d_b - 1$, and thus $d_b = d_a$. ☐

Definition 3.2.3 – Antipodal Graph [14]: For a given graph G, the antipodal graph H has the same vertices as G, but two vertices are adjacent in H if and only if they are antipodal in G.

Corollary to Lemma 3.2.2 [14]: The vertices in any connected component in H must have the same degree in G.

Proof. Vertices are adjacent in H if they are antipodal in G, and lemma 3.2.2 states that antipodal vertices in G have the same degree. ☐

Lemma 3.2.4 [14]: The degree of every vertex of G is at least 2.

Proof. First, note that all vertices must have degree of at least 0 since it is impossible for a vertex to have negative degree in a graph. Second, recall that G is connected with some strictly positive girth $g = 2D+1$, so there can be no vertices with a degree of 0. Since G cannot have a vertex with a degree less than or equal to 0, to prove that every vertex has degree at least 2, it is sufficient to prove that no vertices have degree 1.

We prove this lemma by contradiction, assuming that there exists some vertex, a, such that $d_a = 1$. Let's label the only vertex adjacent to a as b. Since the diameter of G is D, all vertices must be a distance of D or less from a. If a vertex were a distance of D from b, then that vertex must be a distance of $D+1$ from a, since ab is a's only adjacency. Since this would violate the diameter of G, we have that all vertices in G must be a distance of $D-1$ or less from b.

Since G has girth $g = 2D+1$, some cycle exists in G. This cycle cannot include a, because $d_a = 1$ and all vertices in a cycle must have degree of at least 2. This cycle also cannot include b, because a vertex in G must have an antipode in order to be contained in a cycle. To see why this is true, recall that G's girth is $g = 2D+1$, meaning any cycle must include at least $2D+1$ edges. If b were contained in some cycle, then because all vertices are a distance of $D-1$ or less from b, we could form a path of length $D-1$ to the vertex D edges away from b on the cycle, producing a smaller cycle of length at most $2D-1$. Since the existence of any cycle containing b would imply the existence of a cycle of length less than the girth of G, b must not be contained in a cycle.

Let c and d be adjacent vertices on some cycle in G. Since all vertices are within a distance of $D-1$ from b, the distance from b to c is at most $D-1$ and the distance from b to d is at most $D-1$. This implies that there is a cycle of $2D-1$ or fewer edges on b, c and d, a contradiction. □

Theorem 3.2.5 [14]: G is regular.

Proof. Let a and b be any two vertices in H and G. By definition 3.2.3, a and b are adjacent in H if only if they are antipodal in G. If a and b are not antipodal in G, then there exists some path from a to b of length less than D. Since all vertices have degree 2 or more by lemma 3.2.4, we can continue this path until it is of length D and we are at some vertex c. Vertex c must be at distance D from a, for if the distance from a to c were less than D, then the existence of this D path from a to c combined with this shortest path from a to c would imply the existence of a cycle of length at most $2D-1$, which is less than the girth of G.

Since $d_c \geq 2$, c has at least one neighbor that is not already on our path from a to c. Since the diameter of G is D, c's neighbor is a distance of at most D from a, and this path from c's neighbor to a must not include any of the vertices (other than a) of the unique path of length D from a to c because that would contradict the girth of G. We thus have a cycle on vertices a, b, c, and c's neighbor. This cycle must be of length $2D+1$ or greater, because of the girth of G. Since we already know that the distance from a to b to c is D, and the distance from c's neighbor to a cannot be more than D, the only way to satisfy both the diameter and girth requirement is if the distance from c's neighbor to a is D, making the length of this $a-b-c$ cycle $2D+1$.

Let us label the vertices in this $a-b-c$ cycle as n_i, where i is the number of edges traversed before reaching that vertex if we begin at vertex a and travel around the cycle in the direction of the shorter path from a to b. With this vertex labelling, we can write the vertices contained in this cycle in a sequence $(a, c, n_{2D}, n_{D-1}, n_{2D-1}, n_{D-2}, ..., b)$ such that each vertex in this sequence is antipodal to the vertex before it. As a concrete example, see Figure 3, where $D = 5$ and the cycle of interest is of length $2(5)+1 = 11$. In this example, the relevant sequence of vertices is $(a, c, n_{10}, n_4, n_9, n_3, n_8, b)$.

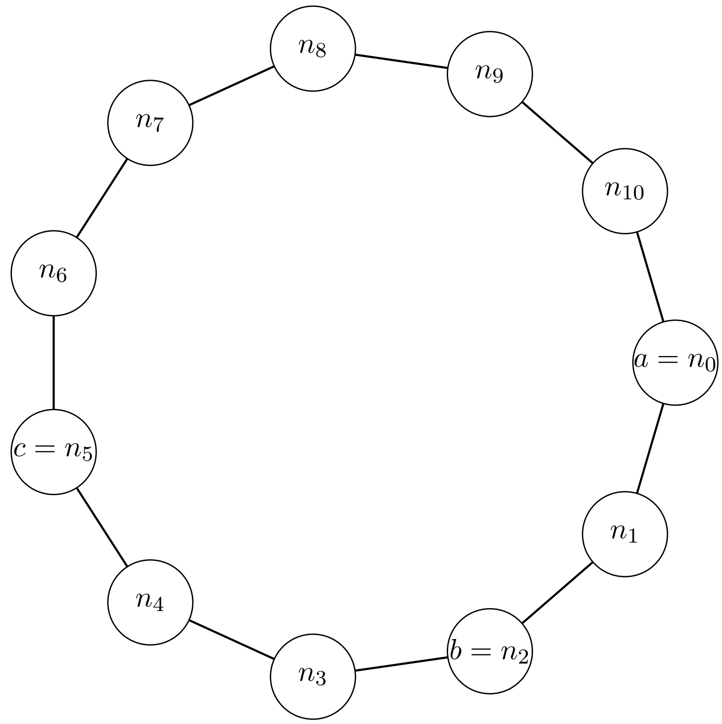


Figure 3: An example of our cycle of interest where $D = 5$.

Such a sequence of vertices, where each vertex in the sequence is antipodal to the vertex before it, must exist. If it did not exist, then at least one pair of vertices that are D away from each other in our cycle would have a path between them of less than D in G, implying the existence of a cycle of length less than $g = 2D + 1$. Since our sequence of vertices $(a, c, n_{2D}, n_{D-1}, n_{2D-1}, n_{D-2}, ..., b)$ is a sequence of antipodal vertices in G, this sequence of vertices forms a path in H connecting a and b.

Since there exists a path in H connecting arbitrary vertices a and b, our corollary to lemma 3.2.2 indicates that a and b have the same degree in G. Since any pair of arbitrary vertices a and b have the same degree in G, we conclude that G is regular; that is, any graph G satisfying definition 3.1.8 of Moore graphs must be regular! □

3.3 Our Three Definitions are Equivalent

In this section, we will prove that our three definitions are equivalent; in other words, any graph that satisfies one of our definitions for a Moore graph must satisfy all three. The result from section 3.2, that graphs satisfying definition 3.1.8 are regular, will be critical to completing this proof. However, because we have only illustrated that graphs satisfying definition 3.1.8 are regular, we must be careful in the way that we wield this information. In the following proof, we show that definition 3.1.4 implies definition 3.1.8 and definition 3.1.7 implies definition 3.1.8, individually. We do not use the fact that G is regular to do so, as we only proved that result for definition 3.1.8. Then we prove that definition 3.1.8 implies definitions 3.1.4 and 3.1.7, using the fact that graphs satisfying definition 3.1.8 are regular as proven in the last section. Since definitions 3.1.4 and 3.1.7 are both equivalent to definition 3.1.8, we then conclude that definitions 3.1.4 and 3.1.7 are equivalent to each other.

Theorem 3.3.1: The three definitions of Moore graphs are equivalent.

Proof.
3.1.4 ⇒ 3.1.8: By definition 3.1.4, G has maximum degree Δ, diameter $\leq D$, and $n_{\Delta,D} = 1 + \Delta(1 + (\Delta - 1) + ... + (\Delta - 1)^{D-1})$ vertices. In order to prove that G satisfies definition 3.1.8, we must show that G has girth $g = 2D + 1$ without using unproven assumptions such as regularity. By scenario 3.1.3, in order for G to achieve the Moore bound, the diameter of G must actually be D, and G must have a subgraph of the form of a tree with branching degree Δ at the root and $\Delta - 1$ at all vertices in levels 0 through $D - 1$. Since the maximum degree of G is Δ, G may have no edges between vertices in levels 0 through $D - 1$ other than those needed to build the tree as specified already. For this reason, any cycle involving the root of the tree must pass through D edges from the root to a leaf of T, as well as D edges from a different leaf back to the root. It may be possible to form cycles of different lengths by traversing a varying number of edges between the vertices in level D, but the smallest possible cycle obviously would involve a single edge. For this reason, the length of any cycle containing the root of the tree subgraph is at least $2D + 1$ edges in length. Since the choice of the root is arbitrary, this fact holds for all vertices in G, meaning that the smallest cycle in G contains at least $2D + 1$ edges. In other words, $g \geq 2D + 1$. Recall from lemma 3.1.9 that a graph G with diameter D has girth $g \leq 2D + 1$. Since we have both $g \geq 2D + 1$ and $g \leq 2D + 1$, we must have $g = 2D + 1$. Thus, G has diameter D and girth $g = 2D + 1$, so G satisfies definition 3.1.8.

3.1.7 ⇒ 3.1.8: By definition 3.1.7, G has minimum degree δ, girth $g = 2k + 1$, and $n_{\delta,g} = 1 + \delta(1 + (\delta - 1) + ... + (\delta - 1)^{k-1})$ vertices. In order to prove that G satisfies definition 3.1.8, we must show that G has diameter D without using unproven assumptions such as regularity. To find G's diameter, recall from scenario 3.1.6 that G has a subgraph of the form of the tree T in figure 2, where the root of T is arbitrary. Since a Moore graph is defined to have the same number of vertices as this subgraph T, there are no vertices in G other than those in T. Since T is a tree of height k, every vertex is a distance of k or fewer edges from the root of T. Furthermore, because the choice of the root of the tree is arbitrary, every vertex is a distance of k or fewer edges from any vertex in G, implying that $D \leq k$. Additionally, because the girth of G is $g = 2k + 1$, there must exist some cycle C in G containing $2k + 1$ vertices. Let u and v be two vertices in C that are separated by k edges on this cycle. If there exists some path of fewer than k edges between u and v in G, that would imply the existence of a cycle of length smaller than the girth of G. For this reason, there must be two vertices u and v in G that are a distance of k from each other, implying that $D \geq k$. Since we have both $D \geq k$ and $D \leq k$, we conclude that $D = k$. Thus G has diameter D and girth $g = 2D + 1$, satisfying definition 3.1.8.

3.1.8 ⇒ 3.1.4 and 3.1.7: By definition 3.1.8, G's diameter is D and G's girth is $g = 2D + 1$. Let G's maximum degree be denoted Δ and minimum degree be denoted δ. By scenario 3.1.3, G has at most $1 + \Delta(1 + (\Delta - 1) + ... + (\Delta - 1)^{D-1})$ vertices. By scenario 3.1.6, and because G's girth is $g = 2D + 1$, G must have at least $1 + \delta(1 + (\delta - 1) + ... + (\delta - 1)^{D-1})$ vertices. Letting n represent the number of vertices in G, these bounds give us the following inequality.

$$1 + \delta(1 + (\delta - 1) + ... + (\delta - 1)^{D-1}) \leq n \leq 1 + \Delta(1 + (\Delta - 1) + ... + (\Delta - 1)^{D-1})$$

Since all graphs fulfilling definition 3.1.8 are regular, we let r represent the degree of all vertices in G.

$$1 + r(1 + (r - 1) + ... + (r - 1)^{D-1}) \leq n \leq 1 + r(1 + (r - 1) + ... + (r - 1)^{D-1})$$

Notice that the expression on both sides of n is the same! This implies that G has exactly $1 + r(1 + (r-1) + ... + (r-1)^{D-1})$ vertices. Since this is the number of vertices required to satisfy both definitions 3.1.4 and 3.1.7, we conclude that 3.1.8 implies both of these definitions.

3.1.4 $\iff$ 3.1.7: By the work above, 3.1.4 $\iff$ 3.1.8 and 3.1.8 $\iff$ 3.1.7. Thus 3.1.4 $\iff$ 3.1.7.

□

Corollary to Theorem 3.3.1: All Moore graphs are regular.

Proof. According to theorem 3.2.5, any graph satisfying definition 3.1.8 of a Moore graph is regular. Since theorem 3.3.1 finds that the three definitions of a Moore graph are equivalent, it is clear that the property of regularity applies to all Moore graphs. □

3.4 Restrictions on Moore Graphs of Girth 5 (Diameter 2)

For this section, let G be a Moore graph with girth $g = 5$ and minimum degree $\delta \geq 3$. By theorem 3.2.5, G is regular, so δ represents the degree of every vertex in G. In this section, we will show that for the nontrivial cases $\delta \geq 3$, there are very few possibilities for the value of δ given $g = 5$. Hoffman and Singleton published the first proof of this result in 1960 using an approach based on the diameter of the graph (which is $D = 2$ for $g = 5$) [8], but our approach will revolve around the girth of G, following the proof Jiří Matoušek includes in *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra* [12].

Lemma 3.4.1 [12]: If G is a Moore graph that is regular of degree $\delta \geq 3$ and has girth $g = 5$, then every pair of non-adjacent vertices have exactly one common neighbor.

Proof. Let u and v be any pair of non-adjacent vertices. Without loss of generality, let u be the root of the tree T as discussed in scenario 3.1.6. Since $g = 2k + 1 = 5$, the height of T is $k = 2$. In order for u and v to not be adjacent, v must be a leaf of T. Since u is the root and v is a leaf of some tree T with height 2 that is a subgraph of G, there exists a path of length 2 connecting u and v in G. Furthermore, this is the unique 2 edge u, v-path in G, for if there were another u, v-path of length 2, then there would be a cycle of length less than G's girth. Since there is a unique 2-edge path from u to v, the vertices u and v have only one common neighbor.

□

Theorem 3.4.2 [12]: If G is a Moore graph that is regular of degree $\delta \geq 3$ and has girth $g = 5$, then $\delta \in \{3, 7, 57\}$.

Proof. Let A be the adjacency matrix of G, and consider $B := A^2$. By the definition of matrix multiplication, we can express each entry of B as follows:

$$b_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

Furthermore, by the definition of adjacency matrix, $a_{ik} = 1$ if vertex i is adjacent to vertex k but is zero otherwise, and $a_{kj} = 1$ if vertex k is adjacent to vertex j but is zero otherwise. By multiplication, the product $a_{ik} a_{kj}$ is:

$$a_{ik} a_{kj} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent to } k, \\ 0 & \text{otherwise.} \end{cases}$$

Since b_{ij} is the sum of $a_{ik}a_{kj}$ for all vertices k in G, we know that for $i \neq j$, b_{ij} represents the number of common neighbors between vertices i and j, and for $i = j$, b_{ij} is the degree of vertex $i = j$.

To further specify B, recall that all vertices in G have degree δ and that every pair of nonadjacent vertices has exactly one common neighbor by lemma 3.4.1. Additionally, every pair of adjacent vertices must have zero common neighbors, as the existence of a common neighbor would imply the existence of a 3-cycle, violating G's girth. With this, we can further specify the entries of matrix B corresponding to G.

$$b_{ij} = \begin{cases} \delta & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are not adjacent,} \\ 0 & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are adjacent.} \end{cases}$$

Knowing this, we can rewrite matrix B as follows.

$$B = A^2 = J_n + (\delta - 1)I_n - A$$

To see why this equation is valid, notice that J_n sets all entries equal to 1; the addition of $(\delta - 1)I_n$ changes only the entries b_{ij} where $i = j$, allowing these entries to equal δ as specified above; and subtracting the adjacency matrix A changes the entries where i and j are adjacent to 0, while leaving nonadjacent vertices, including those on the main diagonal, untouched. Since the right side of the equation satisfies our rules for each entry b_{ij}, this matrix is indeed B.

Since G is δ-regular, there are δ ones and $n - \delta$ zeros in each row of A. For this reason, $A\vec{1} = \delta\vec{1}$, implying that δ is an eigenvalue of A corresponding to eigenvector $\vec{1}$.

Recall from linear algebra that every real and symmetric $n \times n$ matrix has n real but not necessarily distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ corresponding to mutually orthogonal eigenvectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$. Adjacency matrices are real and symmetric, so this fact applies to A. Let us denote $\lambda_1 = \delta$ and $v_1 = \vec{1}$. Because the eigenvectors of A are mutually orthogonal, we have $\vec{1} \cdot \vec{v}_i = \vec{v}_1 \cdot \vec{v}_i = 0$ for all $i \neq 1$, implying $J_n\vec{v}_i = \vec{0}$.

With this in mind, we multiply our equation for $B = A^2$ by any eigenvector $\vec{v}_i$ where $i \neq 1$ and replace A with λ wherever we multiply A by $\vec{v}_i$.

$$A^2 = J_n + (\delta - 1)I_n - A$$
$$A^2\vec{v}_i = J_n\vec{v}_i + (\delta - 1)I_n\vec{v}_i - A\vec{v}_i$$
$$A\lambda_i\vec{v}_i = \vec{0} + (\delta - 1)\vec{v}_i - \lambda_i\vec{v}_i$$
$$\lambda_i^2\vec{v}_i = (\delta - 1)\vec{v}_i - \lambda_i\vec{v}_i$$
$$\lambda_i^2\vec{v}_i = (\delta - 1 - \lambda_i)\vec{v}_i$$

Since both sides of this equation are multiples of $\vec{v}_i$, the coefficients of $\vec{v}_i$ must be equal, implying the following.

$$\lambda_i^2 = (\delta - 1) - \lambda_i$$
$$\lambda_i^2 + \lambda_i - (\delta - 1) = 0$$

The reasoning that lead us to this quadratic equation in terms of λ_i is valid for any eigenvalue-eigenvector pair other than $i = 1$. For this reason, all possible eigenvalues of A other than $\lambda_1 = \delta$ are roots of this quadratic equation. Using the quadratic formula, we find that these these roots, labelled ρ_1 and ρ_2, are as follows.

$$\rho_1 = \frac{-1}{2} + \frac{\sqrt{4\delta - 3}}{2} \text{ and } \rho_2 = \frac{-1}{2} - \frac{\sqrt{4\delta - 3}}{2}$$

To reiterate, we have found that A has 3 eigenvalues: δ, ρ_1, and ρ_2. Since the eigenvalues of every eigenvector other than $\vec{v}_1$ are ρ_1 or ρ_2, the eigenvalue of δ occurs exactly once. Let us suppose that ρ_1 occurs m_1 times and ρ_2 occurs m_2 times. Since there are n eigenvectors, we have that $1 + m_1 + m_2 = n$.

Recall from linear algebra that the sum of all eigenvalues of any $n \times n$ matrix A equals the sum of its diagonal elements, also known as A's trace. Since in our case, A is an adjacency matrix for a graph G without loops, all diagonal entries are zero, implying that the sum of the eigenvalues must be zero. Summing the eigenvalues of A and setting the sum equal to zero, we have the following equation.

$$\delta + m_1\rho_1 + m_2\rho_2 = 0$$

We now substitute in for ρ_1 and ρ_2 and manipulate the equation as follows.

$$\delta + m_1(\frac{-1}{2} + \frac{\sqrt{4\delta - 3}}{2}) + m_2(\frac{-1}{2} - \frac{\sqrt{4\delta - 3}}{2}) = 0$$
$$2\delta - m_1 + m_1\sqrt{4\delta - 3} - m_2 - m_2\sqrt{4\delta - 3} = 0$$
$$(m_1 - m_2)\sqrt{4\delta - 3} = m_1 + m_2 - 2\delta$$

Since $1+m_1+m_2 = n$, we have that $m_1+m_2 = n-1$, allowing us to make another substitution.

$$(m_1 - m_2)\sqrt{4\delta - 3} = n - 1 - 2\delta$$

Here, we note that because G is a Moore graph, by definition 3.1.7, the number of vertices in G is exactly $n_{\delta,5} = 1 + \delta(1 + (\delta - 1) + ... + (\delta - 1)^{k-1})$. As discussed earlier, G's girth of 5 implies $k = 2$, so $n_{\delta,5} = 1 + \delta(1 + (\delta - 1)) = 1 + \delta + \delta^2 - \delta = \delta^2 + 1$ vertices. We can now substitute this expression for n into our equation.

$$(m_1 - m_2)\sqrt{4\delta - 3} = \delta^2 + 1 - 1 - 2\delta$$
$$(m_1 - m_2)\sqrt{4\delta - 3} = \delta^2 - 2\delta$$

In order to find restrictions on the possible values of δ, we first will find restrictions on the possible values of the quantity under the radical, $4\delta - 3$. This quantity is obviously an integer, since vertex degrees are integers and products and differences of integers are also integers. Furthermore, since we are only considering the nontrivial cases $\delta \geq 3$ in this proof, this quantity must be positive. So $4\delta - 3$ is a positive integer. We claim that $4\delta - 3$ is not just a positive integer, but also a perfect square of some other positive integer, s. If $4\delta - 3$ is not the square of some positive integer, then $\sqrt{4\delta - 3}$ is irrational. However, products and differences of integers are integers, forcing the right side of the above equation, $\delta^2 - 2\delta$, to be an integer, and thereby rational. We cannot have one side of our equation rational and the other side irrational, so $\sqrt{4\delta - 3}$ can only be irrational if $m_1 - m_2 = 0$. However, if $m_1 - m_2 = 0$, then $\delta^2 - 2\delta = 0$, which is impossible for $\delta \geq 3$. Thus, $4\delta - 3 = s^2$ for some positive integer s, implying $\delta = \frac{s^2+3}{4}$.

Substituting $\frac{s^2+3}{4}$ for δ and s for $\sqrt{4\delta - 3}$ into the equation we have been working with yields the following.

$$(m_1 - m_2)s = (\frac{s^2 + 3}{4})^2 - 2(\frac{s^2 + 3}{4})$$

$$(m_1 - m_2)s = \frac{1}{16}(s^4 + 6s^2 + 9) - \frac{1}{2}(s^2 + 3)$$
$$16(m_1 - m_2)s = s^4 + 6s^2 + 9 - 8s^2 - 24$$
$$15 = s^4 - 2s^2 - 16(m_1 - m_2)s$$
$$15 = s(s^3 - 2s - 16(m_1 - m_2))$$

This equation implies that 15 is a multiple of our positive integer s. Considering only positive integer factors of 15, we have $s \in \{1, 3, 5, 15\}$. We substitute these possible values of s into our formula for δ, resulting in the following possible values of δ.

$$\text{If } s = 1\text{: } \delta = \frac{1^2 + 3}{4} = 1,$$
$$\text{if } s = 3\text{: } \delta = \frac{3^2 + 3}{4} = 3,$$
$$\text{if } s = 5\text{: } \delta = \frac{5^2 + 3}{4} = 7,$$
$$\text{if } s = 1\text{: } \delta = \frac{15^2 + 3}{4} = 57.$$

Since we assumed $\delta \geq 3$ for our proof, we may exclude $\delta = 1$ from our results. We thus have $\delta \in \{3, 7, 57\}$, concluding our proof. □

3.5 Existence and Uniqueness of the Moore Graph with Diameter 2, Degree 3

In the last section, we proved that nontrivial Moore graphs G with girth $g = 5$ and diameter $D = 2$ cannot exist unless G is regular with vertex degree 3, 7, or 57. In this section, we prove the existence and uniqueness of 3-regular Moore graphs for $D = 2$ and provide part of the setup for the proof of the 7-regular case. The question of existence is still open for the 57-regular case.

Recall that in section 3.2, we relied on definition 3.1.8 of a Moore graph, and in section 3.4, we relied on definition 3.1.7 of a Moore graph. In this section, our approach mirrors that of Hoffman and Singleton's, relying on definition 3.1.4 of a Moore graph. Our use of three different definitions for three different objectives illustrates the utility of having several equivalent definitions for the same idea.

Since we rely on definition 3.1.4 in this section, let the degree of every vertex in G be denoted Δ. In order to prove the existence and uniqueness of these graphs, we need to construct the adjacency matrix A of G. To specify this adjacency matrix, we must decide how we will label our vertices. We let 0 be any arbitrary vertex, and we think of 0 as the root of the subgraph T as illustrated in Figure 2. Since $D = 2$ and T contains all of G's vertices, we can think of G as having 3 levels, called level 0, level 1, and level 2.

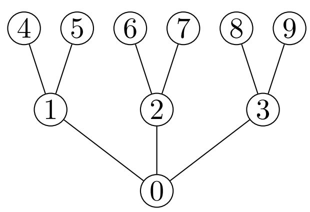
- Level 0 consists exclusively of vertex 0, the root of our subgraph T.
- Level 1 consists of the Δ adjacencies of 0, which we label $1, 2, ..., \Delta$ in arbitrary order.
- Level 2, the final level, consists of all vertices other than 0 that are adjacent to vertices in level 1. For notational clarity, we denote the subset of vertices in level 2 that are adjacent to vertex $i \in \{1, 2, ..., \Delta\}$ as S_i. We label the vertices of S_1 as $\Delta + 1$ through $2\Delta - 1$ in arbitrary order within S_1. Similarly, we label the vertices of S_2 as 2Δ through $3\Delta - 2$ in arbitrary order within S_2, and so on such that the vertices of S_i are labelled as $i(\Delta - 1) + 2$ through $i(\Delta - 1) + \Delta$ in arbitrary order within S_i.

With this vertex labelling, we are now prepared to construct the adjacency matrix of A.

Construction 3.5.1 [8]: The adjacency matrix of a Moore graph with $\Delta = 3$.

We will start by constructing the adjacency matrix for the more manageable case of $\Delta = 3$, which we denote $A_{\Delta=3}$. To construct the adjacency matrix $A_{\Delta=3}$, we let the rows and columns be indexed such that the first row and first column depict the adjacencies of vertex 0, the second row and second column depict the adjacencies of vertex 1, etc. Additionally, we use horizontal and vertical lines to distinguish between the levels of G; within level 2, we also use lines to distinguish between the S_i. The entries above the first horizontal line (equivalently, the entries to the left of the first vertical line) are the adjacencies of level 0. The entries between the first and second horizontal lines (equivalently, the entries between the first and second vertical lines) are the adjacencies of the vertices in level 1. The entries in the remaining blocks correspond to the adjacencies of vertices in level 2, with separating lines between the adjacencies of S_1, S_2, ..., S_Δ.

With this organizational scheme for our matrix in mind, we construct $A_{\Delta=3}$ below, primarily by referencing subgraph T of G, first shown in Figure 2. We include a reproduced version of this subgraph below with our specified vertex labelling. In $A_{\Delta=3}$, note that row indices are labelled to the left of the matrix and the number of rows in each block is labelled to the right of the matrix.



$$A_{\Delta=3} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \left[\begin{array}{c|ccc|cc|cc|cc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
0 & 1 & 0 & 0 & 0 & 0 & \multicolumn{2}{c|}{P_{12}} & \multicolumn{2}{c}{P_{13}} \\
0 & 1 & 0 & 0 & 0 & 0 & & & & \\ \hline
0 & 0 & 1 & 0 & \multicolumn{2}{c|}{P_{21}} & 0 & 0 & \multicolumn{2}{c}{P_{23}} \\
0 & 0 & 1 & 0 & & & 0 & 0 & & \\ \hline
0 & 0 & 0 & 1 & \multicolumn{2}{c|}{P_{31}} & \multicolumn{2}{c|}{P_{32}} & 0 & 0 \\
0 & 0 & 0 & 1 & & & & & 0 & 0
\end{array}\right] \begin{array}{l} 1 \\ \\ \Delta = 3 \\ \\ \\ \Delta - 1 = 2 \\ \\ \Delta - 1 = 2 \\ \\ \Delta - 1 = 2 \end{array}$$

Because T includes all vertices of G as well as all edges other than those within level 2, all entries of this matrix $A_{\Delta=3}$ are completely determined from the subgraph T except for entries corresponding to edges between vertices in level 2. In order to fill in these missing entries, we need to deduce a few results.

First, we consider whether there can be edges between vertices within a given S_i. Well, the existence of such an edge would obviously imply the existence of 3-cycle. Since a Moore graph with diameter 2 has girth 5, a 3-cycle is not possible in G, prohibiting the existence of edges between vertices within a given S_i. For this reason, the blocks that could be labelled P_{11}, P_{22}, and P_{33} are completely filled with zeros.

At this point, we have specified every entry of $A_{\Delta=3}$ except for those in the principal submatrices labelled P_{ij}. These principal submatrices correspond to the adjacencies between vertices in different S_i and S_j. We will discuss these eventually, but at this juncture it makes sense to construct a more general adjacency matrix that works for any $\Delta \in \{3, 7, 57\}$.

Construction 3.5.2 [8]: The adjacency matrix of a Moore graph with $\Delta \in \{3, 7, 57\}$.

We included the previous construction because is easier to understand how an adjacency matrix arises when we can visualize both the graph and the matrix. However, if we wanted to establish the existence and uniqueness of Moore graphs with both $\Delta = 3$ and $\Delta = 7$, a generalized adjacency matrix would be useful. Wielding a more general form, we can prove statements relevant to both cases simultaneously, avoiding repetition and redundancy.

Following the exact same reasoning we used in the case of $\Delta = 3$, we construct the following adjacency matrix A for any Moore graph of arbitrary vertex degree Δ. The row index is labelled to the left and the number of rows in each block is labelled to the right of the matrix.

$$A = \begin{array}{r} 0 \\ 1 \\ 2 \\ \vdots \\ \Delta \\ \Delta+1 \\ \Delta+2 \\ \vdots \\ 2\Delta-1 \\ 2\Delta \\ 2\Delta+1 \\ \vdots \\ 3\Delta-2 \\ \vdots \\ \Delta(\Delta-1)+2 \\ \Delta(\Delta-1)+3 \\ \vdots \\ \Delta(\Delta-1)+\Delta \end{array} \left[\begin{array}{c|cccc|cccc|cccc|c|cccc}
0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline
1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
0 & 1 & 0 & \cdots & 0 & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\
0 & 1 & 0 & \cdots & 0 & \multicolumn{4}{c|}{0} & \multicolumn{4}{c|}{P_{12}} & \cdots & \multicolumn{4}{c}{P_{1\Delta}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\
0 & 1 & 0 & \cdots & 0 & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\ \hline
0 & 0 & 1 & \cdots & 0 & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\
0 & 0 & 1 & \cdots & 0 & \multicolumn{4}{c|}{P_{21}} & \multicolumn{4}{c|}{0} & \cdots & \multicolumn{4}{c}{P_{2\Delta}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\
0 & 0 & 1 & \cdots & 0 & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\ \hline
\vdots & \multicolumn{4}{c|}{\vdots} & \multicolumn{4}{c|}{\vdots} & \multicolumn{4}{c|}{\vdots} & \ddots & \multicolumn{4}{c}{\vdots} \\ \hline
0 & 0 & 0 & \cdots & 1 & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\
0 & 0 & 0 & \cdots & 1 & \multicolumn{4}{c|}{P_{\Delta 1}} & \multicolumn{4}{c|}{P_{\Delta 2}} & \cdots & \multicolumn{4}{c}{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{} \\
0 & 0 & 0 & \cdots & 1 & \multicolumn{4}{c|}{} & \multicolumn{4}{c|}{} & & \multicolumn{4}{c}{}
\end{array}\right] \begin{array}{l} 1 \\ \\ \\ \Delta \\ \\ \\ \\ \Delta-1 \\ \\ \\ \Delta-1 \\ \\ \\ \Delta-1 \\ \\ \\ \Delta-1 \\ \\ \end{array}$$

Just as in the $\Delta = 3$ case, all entries of A are completely determined from the subgraph T in scenario 3.1.3 except for entries corresponding to edges between vertices in level 2. Also just as in the $\Delta = 3$ case, the existence of an edge between two vertices within a given S_i would imply the existence of 3-cycle. Since G has girth $g = 5$, a 3-cycle is not possible in G, so there are no edges between vertices within a given S_i, resulting in the blocks that could be labelled $P_{11}, P_{22}, ..., P_{\Delta\Delta}$ being completely filled with zeros.

With this matrix, we have specified every entry of A except for those in the principal submatrices labelled P_{ij}. These principal submatrices correspond to the adjacencies between vertices in different S_i and S_j, which we call *reentering arcs*. Determining the structure of a Moore graph with $\Delta \in \{3, 7, 57\}$ and hence proving the existence of a given Moore graph amounts to determining the entries in these submatrices. We begin to characterize these submatrices in the following theorem.

Theorem 3.5.3 [8]: The P_{ij} blocks in A are permutation matrices.

Proof. For $i, j \in \{1, 2, ..., \Delta\}$ such that $i \neq j$, let a be any vertex in S_i and let b and c be any two distinct vertices in S_j. Vertex a cannot be adjacent to both b and c, for that would imply the existence of a 4-cycle on vertices j, b, a, and c, a violation of G's girth. Since every vertex in level 2 is contained in some S_i and there are no adjacencies between vertices within S_i, this implies that no vertex in level 2 may be adjacent to more than 1 vertex in any S_j.

Recall that each vertex in level 2 is adjacent to exactly one vertex in level 1 and no vertices in level 0. Because G is Δ-regular, these constraints force every vertex in level 2 to be adjacent to $\Delta - 1$ other vertices in level 2 via reentering arcs. There are $\Delta - 1$ subsets other than the subset containing a vertex, so any vertex in subset S_i of level 2 is adjacent to exactly one vertex in every other subset of level 2. For this reason, every row and every column of each P_{ij} contains exactly one 1; all other entries of each P_{ij} are 0. By definition 2.2.1, each P_{ij} is a permutation matrix. □

Theorem 3.5.4 [8]: We can number the vertices within each S_i in level 2 such that $P_{1i} = P_{i1} = I$.

Proof. In constructing A, we did not assign any particular order within each S_i in level 2. Consider an arbitrary ordering of the vertices in S_1. As illustrated in the proof of theorem 3.5.3, each vertex in S_1 is adjacent to exactly 1 vertex in all of the other S_i's, and no two vertices in S_1 may be adjacent to the same vertex in any S_i. If the vertices in each S_i are ordered in ascending order according to the label of their adjacent vertices in S_1, then $P_{1i} = I$ and $P_{i1} = I$ for each $i \in \{2, 3, ..., \Delta\}$. □

Definition 3.5.5 – Canonical Form [8]: Let G be a Moore graph with diameter $D = 2$. The adjacency matrix of G is in its canonical form if it has the form of A with the additional arrangement from theorem 3.5.4 that $P_{1i} = P_{i1} = I$ for each $i \in \{2, 3, ..., \Delta\}$.

Notation 3.5.6 [8]: In the subsequent lemmas and theorems, it is useful for us to discuss the adjacencies between vertices within level 2. Notice that all of these adjacencies occur in the lower right of our adjacency matrix A in the blocks of 0's and P_{ij}'s. To facilitate discussion of this part of our adjacency matrix, we denote this submatrix with the letter B.

Notation 3.5.7 [8]: Consider a square matrix of $\Delta \times \Delta$ blocks, each of which is a $\Delta - 1 \times \Delta - 1$ matrix. We let K be a matrix of this form, where the blocks on the main diagonal are entirely 1's and all other blocks are entirely 0's. In the $\Delta = 3$ case, K would have the following form.

$$K_{\Delta=3} = \left[\begin{array}{cc|cc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Lemma 3.5.8 [8]: The following equation holds for any Moore graph G of diameter $D = 2$.

$$B^2 = J - K + (\Delta - 1)I - B$$

Proof. Recall that for Moore graphs of girth $g = 5$ and diameter $D = 2$, we found the following equation to be true in section 3.4.

$$A^2 = J_n + (\Delta - 1)I_n - A$$

Here, we employ similar logic to find a related expression for the principal submatrix B. Consider $C := B^2$. By the definition of matrix multiplication, we can express each entry of C as follows:

$$c_{ij} = \sum_k b_{ik} b_{kj}$$

Furthermore, by the definition of adjacency matrix, $b_{ik} = 1$ if vertex i is adjacent to vertex k but is zero otherwise, and $b_{kj} = 1$ if vertex k is adjacent to vertex j but is zero otherwise. By multiplication, the product $b_{ik} b_{kj}$ is:

$$b_{ik}b_{kj} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent to } k, \\ 0 & \text{otherwise.} \end{cases}$$

Since c_{ij} is the sum of $b_{ik}b_{kj}$ for all vertices k in level 2 of G, we know that for $i \neq j$, c_{ij} represents the number of common neighbors between vertices i and j within level 2, and for $i = j$, c_{ij} is the number of adjacencies of vertex $i = j$ within level 2.

To further specify C, recall that all vertices in level 2 are adjacent to $\Delta - 1$ other vertices in level 2 as discussed in the proof of theorem 3.5.3. Additionally, lemma 3.4.1 states that every pair of nonadjacent vertices have exactly one common neighbor in G. For distinct vertices i and j within an S_l, that common neighbor is obviously vertex l, so vertices i and j within an S_l have no common neighbors within level 2 (also shown in a different way in the proof of theorem 3.5.3). On the other hand, for nonadjacent vertices i and j in different S_l's, that common neighbor must be somewhere in level 2, since no vertices in any S_l are adjacent to any vertex in level 1 other than l, nor are they adjacent to the vertex in level 0. Finally, adjacent vertices i and j in level 2 have 0 common neighbors, as that would imply the existence of a 3-cycle, contradicting G's girth.

With this, we can further specify the entries of matrix C.

$$c_{ij} = \begin{cases} \Delta - 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \text{ but } i \text{ and } j \text{ are within the same } S_l, \\ 1 & \text{if } i \neq j,\ i \text{ and } j \text{ are within different } S_l\text{'s, but } i \text{ and } j \text{ are not adjacent,} \\ 0 & \text{if } i \neq j,\ i \text{ and } j \text{ are within different } S_l\text{'s, but } i \text{ and } j \text{ are adjacent.} \end{cases}$$

Knowing this, we can rewrite matrix C as follows. Note that all matrices in this equation are $\Delta(\Delta - 1) \times \Delta(\Delta - 1)$ square matrices.

$$C = B^2 = J - K + (\Delta - 1)I - B$$

To see why this equation is valid, notice that J initially sets all entries equal to 1. Since c_{ij} represents the number of common neighbors between vertices i and j within level 2, subtracting K adjusts the matrix to account for the fact that there are no common neighbors within a given S_l. However, when $i = j$, c_{ij} is actually the number of adjacencies of vertex $i = j$ within level 2. Adding $(\Delta - 1)I$ thus corrects for this subtraction, allowing the diagonal entries to all be $(\Delta - 1)$, the number of adjacencies within level 2 of each vertex in level 2. At this point, the diagonal blocks of B^2 are correct, but every entry outside of the diagonal blocks is set to 1. By subtracting the principal submatrix B, we change the entries where i and j are in different S_l's and are adjacent from 1 to 0, while leaving nonadjacent vertices, including those in the diagonal blocks, untouched. Since the right side of the equation satisfies our rules for each entry c_{ij}, this matrix is indeed C. □

Theorem 3.5.9 [11]: Let X and Y be two partitioned matrices. If (1) the number of columns in X equals the number of rows in Y and (2) the column partition of X is the same as the row partition of Y, then we may compute the matrix product XY as if each block were itself a scalar.

This is a result from linear algebra which can be readily found in a linear algebra textbook.

Lemma 3.5.10 [8]: If G is a Moore graph with diameter $D = 2$ and $i \neq j$, then the following equation is true.

$$\sum_k P_{ik}P_{kj} + P_{ij} = J_\Delta$$

Proof. First, we seek to perform block multiplication of B with itself. Note that the first condition of theorem 3.5.9 is satisfied because B is a square matrix, and the second condition of theorem 3.5.9 is satisfied because B is composed of $\Delta \times \Delta$ blocks, each of which is a $\Delta - 1 \times \Delta - 1$ matrix. Since B satisfies both conditions of theorem 3.5.9 for block multiplication with itself, we can compute the matrix product B^2 as though each block of B were itself a scalar.

By the definition of matrix multiplication, the entry at the i^{th} row of blocks and the j^{th} column of blocks of B^2 can be expressed as follows.

$$\sum_k P_{ik} P_{kj}$$

Recall the following equation from lemma 3.5.8.

$$B^2 = J - K + (\Delta - 1)I - B \Rightarrow B^2 - (\Delta - 1)I + B = J - K$$

Consider any P_{ij} in B such that $i \neq j$ as well as the corresponding submatrix in each of the matrices of the equation above. Since $i \neq j$, we have that $(\Delta - 1)I$ and K are exclusively zeros in the submatrix of interest. For this reason, the equation above reduces to the following in the block we are considering.

$$\sum_k (P_{ik} P_{kj}) + P_{ij} = J_\Delta$$

□

Theorem 3.5.11 [8]: There exists a unique Moore graph with diameter $D = 2$ and regular vertex degree $\Delta = 3$: the Petersen graph.

Proof. Let $G_{\Delta=3}$ be a Moore graph of diameter $D = 2$ with regular vertex degree $\Delta = 3$, and let G's adjacency matrix $A_{\Delta=3}$ be in its canonical form. By definition 3.5.5, $A_{\Delta=3}$ is structured that $P_{1i} = P_{i1} = I$, leading B to have the following form.

$$B = \left[\begin{array}{c|c|c} 0 & I & I \\ \hline I & 0 & P_{23} \\ \hline I & P_{32} & 0 \end{array}\right]$$

Since $\Delta = 3$, P_{23} and P_{32} are 2×2 matrices. Furthermore, by theorem 3.5.3, P_{23} and P_{32} are permutation matrices. Recall from lemma 3.5.10 that the following result is true if $i \neq j$.

$$\sum_k (P_{ik} P_{kj}) + P_{ij} = J_\Delta$$

Since $2 \neq 3$, that result must hold for P_{23} and P_{32}. If P_{23} were I_2, then $\sum_k (P_{2k} P_{k3})$ would equal I_2. Substituting I_2 for P_{23} and $\sum_k (P_{2k} P_{k3})$ into the result from lemma 3.5.10, we have the following.

$$\sum_k (P_{2k} P_{k3}) + P_{23} = 2I_2 = J_2$$

This equation is obviously false. Failing to satisfy lemma 3.5.10, we conclude that $P_{23} \neq I_2$. The same reasoning leads us to conclude that $P_{32} \neq I_2$.

There are only two possible 2×2 permutation matrices, so in order for G to exist, we must have the following.

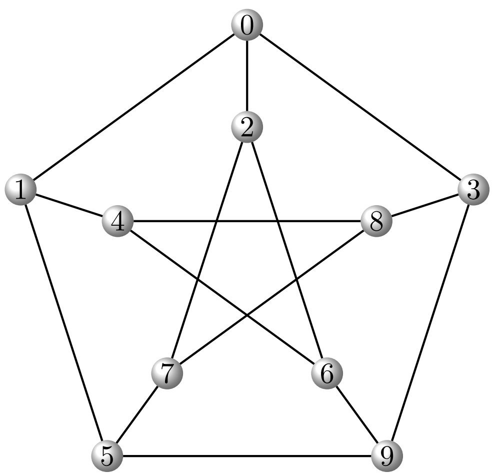
$$P_{23} = P_{32} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Notice that this selection of matrices satisfies the lemma that all other possibilities could not.

$$\sum_k (P_{2k}P_{k3}) + P_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = J_2$$

$$\sum_k (P_{3k}P_{k2}) + P_{32} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = J_2$$

Since any Moore graph with $D = 2$ and $\Delta = 3$ must have an adjacency matrix of the form $A_{\Delta=3}$, and there is only one valid choice of P_{23} and P_{32}, there is a unique Moore graph with $D = 2$ and $\Delta = 3$. Because we have determined the structure of these remaining two submatrices, we can finally complete the adjacency matrix $A_{\Delta=3}$, which was first presented in construction 3.5.1. Below, we include the completed adjacency matrix as well as the resulting graph.



$$\left[\begin{array}{c|ccc|cc|cc|cc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}\right]$$

Notice that the unique Moore graph of diameter $D = 2$ and regular vertex degree $\Delta = 3$ is the Petersen graph! By inspection, one can also verify that the Petersen graph satisfies all 3 of our definitions of a Moore graph. □

3.6 Do Other Moore Graphs Exist?

To some readers, it may seem strange that this paper focused so heavily on the very specific case of Moore graphs with diameter 2, girth 5, and regular degree 3. A rigorous proof of the existence or nonexistence of other Moore graphs was too long to include in this paper, but we provide a brief overview of the major known results here. In Hoffman and Singleton's paper, they prove the existence and uniqueness of the Moore graph with diameter 2, girth 5, and regular degree 7 [8]. In section 3.5, we provide some set-up for their proof by constructing a general adjacency matrix for arbitrary vertex degree Δ and deducing some knowledge about several of this adjacency matrix's principal submatrices, but for the full proof of this result, please see their paper. Interestingly, their solution, known as the Hoffman-Singleton graph, is obtained by piecing together many copies of the Petersen graph [12].

Surprisingly, there are no other known nontrivial Moore graphs, and it is very possible that no others exist. For diameter $D = 3$, Hoffman and Singleton showed that there are no Moore graphs with degree greater than 2 [8]. Furthermore, in 1973, one paper by Bannai and Ito and another by Damerell independently proved that there exist no Moore graphs with diameter $D \geq 4$ [1]. Thus, the question of existence and uniqueness of Moore graphs has been settled for all cases except for the one remaining possibility of a graph with diameter $D = 2$, girth $g = 5$, and regular vertex degree $\Delta = 57$. The task of proving or disproving the existence of such a graph remains an open problem and an active area of research to this day [9].

4 Conclusion

In this paper, we witnessed how representing graphs with adjacency matrices allows us to apply the powerful machinery of linear algebra to graph theoretic problems. First we saw an elegant proof of the fact that K_{10} cannot be decomposed into three copies of the Petersen graph. This proof relied on results from linear algebra, illustrating the power of eigenvalues and vector spaces in graph theory. Next, we discussed three different definitions of Moore graphs, showing that all Moore graphs are regular and all three of our definitions are equivalent. Finally, we proved that Moore graphs with a girth of 5 and a diameter of 2 can only be $3, 7$, or 57 regular, and we proved the existence and uniqueness of the 3-regular case.

Overall, this paper introduced readers to the field of spectral graph theory by highlighting two prominent and beautiful examples. We hope we leave readers with an appreciation for spectral methods and an eagerness to learn more.

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